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OPTIMAL PACKINGS OF K_4's INTO A K_n - THE CASE $n \not\equiv 2 \pmod 3$

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Optimal packings of K_4's into a K_n - The case $n \not\equiv 2 \pmod 3$.

by

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ABSTRACT

In this paper we construct a pairwise balanced design $B(\{4,7^*\},1;n)$ (i.e. a design with blocks of size 4 or 7 and exactly one block of size 7, on n points with $\lambda = 1$) for each $n \equiv 7$ or $10 \pmod{12}$ except $n = 10$ or 19 (in which cases such a design cannot exist). From these designs optimal packings of K_4's into a K_n are derived for $n \not\equiv 2 \pmod 3$, $n \notin \{9,10,18,19\}$, while the case $n \in \{9,10,18\}$ is treated by ad hoc methods. It is not known whether the known packing of 25 K_4's in K_{19} is optimal.

KEY WORDS & PHRASES: *pairwise balanced design, scarce design, packing, constant weight code*

1. INTRODUCTION

Let I_n be a finite set of n elements. For $n \geq k \geq t$ let $D(n,k,t)$ be the largest integer b such that there exist b subsets $B_1,\dots,B_b$ of I_n, each of k elements, such that every t-element subset of I_n is contained in at most one of them.

In a previous paper ([1]) the present author and A. Schrijver determined $D(n,4,2)$ for $n \equiv 2 \pmod 6$. Here we treat $n \equiv 0$ or $1 \pmod 3$ (except $n = 19$), and in a future paper we will discuss the remaining case $n \equiv 5 \pmod 6$. The overall result is the following:

Define

$$J(n,4,2) = \begin{cases} \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12} \\ \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor & \text{otherwise.} \end{cases}$$

Then

(i) for each n $D(n,4,2) \leq J(n,4,2)$
(this is the so-called Johnson bound, see e.g. JOHNSON [4])

(ii) for almost all n $D(n,4,2) = J(n,4,2)$.

Cases in which $D(n,4,2) \neq J(n,4,2)$ is known:

n	8	9	10	11	17	19
J(n,4,2)	4	4	6	8	21	27
D(n,4,2)	2	3	5	6	20	25 or 26

I conjecture that for all other n equality holds. (In any case all further exceptions must have $n \equiv 11 \pmod{12}$).

<u>REMARK</u>. The undefined notations, especially for various types of designs, are taken from HANANI ([3]) or WILSON ([10]).

2. OPTIMAL PACKINGS

A. <u>The case $n \equiv 0,1,3$ or $4 \pmod{12}$</u>

For $n \equiv 1$ or $4 \pmod{12}$ HANANI ([2]) has constructed a Steiner system $S(2,4,n)$. Therefore $D(n,4,2) = J(n,4,2) = \frac{1}{12}n(n-1)$ for these n. Throwing away a fixed point and all blocks containing it produces a system of $\frac{1}{12}n(n-1) - \frac{1}{3}(n-1) = \frac{1}{12}(n-1)(n-4)$ four-tuples on n-1 points, i.e. $D(n,4,2) = J(n,4,2) = \frac{1}{12}n(n-3)$ for $n \equiv 0$ or $3 \pmod{12}$.

B. <u>The case $n \equiv 6,7,9$ or $10 \pmod{12}$</u>

For $n \equiv 7$ or $10 \pmod{12}$, $n \neq 10,19$, we will construct in the next section a pairwise balanced design on n points with $\lambda = 1$ and blocks of size 4 or 7, using exactly one block of size 7 (notation: $B(\{4,7^*\},1;n)$). If we replace the block $\{x_0,\ldots,x_6\}$ of size 7 of such a design by the two four-tuples $\{x_0,x_1,x_2,x_3\}$ and $\{x_0,x_4,x_5,x_6\}$ we have a collection of $\frac{1}{6}(\binom{n}{2}-\binom{7}{2}) + 2 = \frac{1}{12}(n(n-1)-18) = J(n,4,2)$ four-tuples without a common pair.

Hence $D(n,4,2) = J(n,4,2) = \frac{1}{12}(n(n-1)-18)$ for $n \equiv 7$ or $10 \pmod{12}$, $n \neq 10,19$.

Throwing away one point (from the set $\{x_1,\ldots,x_6\}$) yields:

$$D(n,4,2) = J(n,4,2) = \frac{1}{12}(n(n-3)-6) \text{ for } n \equiv 6 \text{ or } 9 \pmod{12}, \quad n \neq 9,18.$$

For the exceptional cases we have

$$D(9,4,2) = 3$$

and

$$D(10,4,2) = 5$$

as can be immediately verified. Next

$$D(18,4,2) = 22$$

as follows from packings constructed by S. Lin and H.R. Phinney.

We give here the packing of H.R. Phinney since it has the largest automorphism group (sc. $\mathbb{Z}_2$, generated by π := (0 10)(1 9)(2 13)(3 4)(5)(6 15)(7 14)(8 17)(11)(12)(16)).

0	1	2	3	1	10	14	17	3	9	12	17
0	4	5	6	2	4	15	17	4	9	10	13
0	7	8	9	2	5	11	13	5	7	12	14
0	10	11	12	2	6	7	10	6	9	11	14
0	13	14	15	2	8	14	16	6	12	15	16
1	4	8	12	3	4	11	16	7	13	16	17.
1	5	9	16	3	5	10	15				
1	7	11	15	3	6	8	13				

The value of D(19,4,2) is not yet known; as a lower bound we have D(19,4,2) $\geq$ 25 as follows from a packing constructed by H.R. Phinney (which is given below) while on the other hand D(19,4,2) $\leq$ 26 as we shall prove below.
First the design:

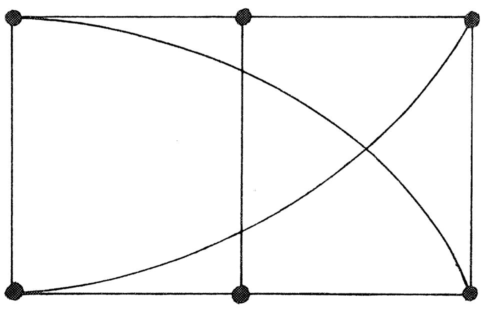
0	1	2	3	1	7	15	17	3	8	11	15
0	4	5	6	1	10	14	18	4	9	10	15
0	7	8	9	2	4	13	17	5	7	11	14
0	10	11	12	2	5	15	18	5	8	10	13
0	13	14	15	2	6	7	10	6	8	14	17
0	16	17	18	2	9	12	14	6	12	15	16
1	4	8	12	3	4	7	16	7	12	13	18.
1	5	9	16	3	5	12	17				
1	6	11	13	3	6	9	18				

PROPOSITION. *If* D(n,4,2) = J(n,4,2) *for some* n $\equiv$ 7 or 10 (mod 12) *then the edges not covered by a maximal packing of* K_4's *into* K_n *form a regular graph on* 6 *points with valency* 3.

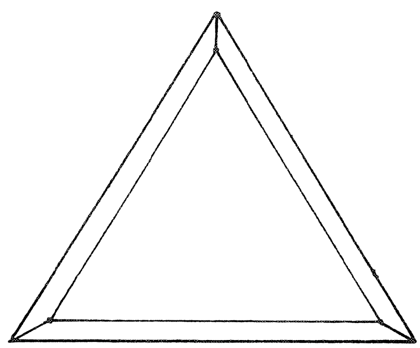
PROOF. $J(n,4,2) = \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1 = \frac{1}{12}(n(n-1)-18)$.

Each quadruple covers six edges, hence J(n,4,2) quadruples cover all edges except nine. Let G be the graph (without isolated vertices) formed by these nine edges. In K_n each point has valency $n-1 \equiv 0 \pmod 3$, and each quadruple removes 0 or 3 edges incident with a given point, hence in G each point has valency $\equiv 0 \pmod 3$. Clearly valency ≥ 9 is impossible. If some point p in G has valency 6, then its 6 neighbours need at least 6 other edges in order to reach valency 3 each; but there are only nine edges in all, so valency 6 does not occur and G is regular, and hence has 6 vertices. □.

LEMMA. *There are only two graphs on 6 points, regular with valency 3:* $K_{3,3}$ *and the prisma:*



and



□

PROPOSITION. D(19,4,2) ≠ J(19,4,2).

PROOF. The edges of both graphs mentioned in the previous lemma can be covered with 3 K_4's. Therefore if D(n,4,2) = J(n,4,2) then C(n,4,2) ≤ J(n,4,2) + 3 (where C(n,4,2) is the number of K_4's necessary to cover all edges of K_n). But J(19,4,2) = 27 and MILLS ([7]) proved that C(19,4,2) = 31 (by exhaustive computer search). Hence D(19,4,2) ≤ 26. □.

3. THE CLASS $\mathbb{B}\{4,7^*\}$

Let $\mathbb{B}\{4,7^*\}$ be the set of integers n for which there exists a pairwise balanced design on n points with blocks of size 4 or 7 and exactly one block of size 7 (and $\lambda=1$). Then

<u>THEOREM</u>. $\mathbb{B}\{4,7^*\} = \{n \mid n \equiv 7 \text{ or } 10 \pmod{12}\} \setminus \{10,19\}$.

Since by Hanani $\mathbb{B}\{4\} = \{n \mid n \equiv 1 \text{ or } 4 \pmod{12}\}$ we have as an immediate corollary:

<u>COROLLARY</u>. [Wilson] $\mathbb{B}\{4,7,10,19\} = \{n \mid n \equiv 1 \pmod 3\}$.

<u>PROOF OF THE THEOREM</u>. Suppose $n \in \mathbb{B}\{4,7^*\}$. By considering the valency of a point it follows that $n \equiv 1 \pmod 3$. Next, since $\binom{4}{2} = 6$ is even and $\binom{7}{2} = 21$ is odd, it follows that $\binom{n}{2}$ must be odd, so that $n \equiv 7$ or $10 \pmod{12}$. Also we saw in the previous section that $n \notin \{10,19\}$. [Note: this argument used that $D(19,4,2) \neq J(19,4,2)$ which is difficult to verify; on the other hand it is easy to see that if $n \in \mathbb{B}(K)$, where K is minimal (each element of K is used as a block size), then $n \geq (\min K-1).\max K + 1$. In our case this means that $n \geq (4-1).7+1 = 22$.] This proves the easy half of the theorem; the remainder of this section is devoted to the other half.

(i) The Truncated Transversal Design.

<u>LEMMA 1</u>. [Truncated Transversal] *If* $\{3t+7,3h+7\} \subset \mathbb{B}\{4,7^*\}$ *and* $t \geq h$ *then* $12t + 3h + 7 \in \mathbb{B}\{4,7^*\}$.

<u>PROOF</u>. As usual: take a transversal design $T(5,1;t)$ (which exists since $t \equiv 0$ or $1 \pmod 4$) and throw away $t-h$ points of one group. This leaves a design with blocks of size 4 or 5 and groups of size h or t on a set X with $|X| = 4t+h$. Next split each point into three points, constructing group-divisible designs $GD(4,1,3)$ on the sets of size $3\times4 = 12$ and $3\times5 = 15$, that is, make a design on the set $X\times I_3$ by taking for each group G of the original design a new group $G\times I_3$, and for each block B the blocks of a $GD(4,1,3;3|B|)$ constructed in such a way that it has groups $\{b\}\times I_3$. We now have a design with blocks of size 4 and groups of size $3h$ or $3t$. Adding a block Z of 7 points and the designs (on the sets $(G\times I_3)\cup Z$) $B(\{4,7^*\},1;3h+7)$ and $B(\{4,7^*\},1;3t+7)$ which exist by hypothesis, we obtain the required design $B(\{4,7^*\},1;12t+3h+7)$. □.

Let $x \equiv 7$ or $10 \pmod{12}$. There are 8 cases mod 48:

For $x \equiv 7$ or $19 \pmod{48}$ write $x = 12t + 7$ ($h=0$, $t\equiv0$ or $1 \pmod 4$).
If we assume that $3t + 7 \in \mathbb{B}\{4,7^*\}$ then $x \in \mathbb{B}\{4,7^*\}$ follows.
We may do this except for $t = 1$ or 4, hence we get x unless $x = 19$ or 55.
$19 \notin \mathbb{B}\{4,7^*\}$, and 55 will be done later.

For $x \equiv 22$ or $34 \pmod{48}$ write $x = 12t + 3.5 + 7$, ($h=5$, $t\equiv 0,1 \pmod 4$)). If we assume that $3t + 7 \in \mathbb{B}\{4,7^*\}$ and $t \geq 5$ then $x \in \mathbb{B}\{4,7^*\}$ follows. We still have to do 22, 34 and 70.

For $x \equiv 31$ or $43 \pmod{48}$ write $x = 12t + 3.8 + 7$, ($h=8$, $t\equiv 0,1 \pmod 4$)). Again for $t \geq 8$ $x \in \mathbb{B}\{4,7^*\}$ follows provided that we can do 31,43,79 and 91.

For $x \equiv 46 \pmod{48}$ write $x = 12t + 3.9 + 7$, ($h=9$, $t \equiv 1 \pmod 4$)), this yields $x \geq 142$. We still have to do 46 and 94.

For $x \equiv 10 \pmod{48}$ write $x = 12t + 3.13 + 7$, ($h=13$, $t \equiv 1 \pmod 4$)), this yields $x \geq 202$. We still have to do 58, 106 and 154.

Therefore the theorem will be proved if we show that

$$\{22,31,34,43,46,55,58,70,79,91,94,106,154\} \subset \mathbb{B}\{4,7^*\}.$$

(ii) Kirkman Designs.

<u>LEMMA 2</u>. *For each* t: $9t + 4 \in \mathbb{B}\{4,(3t+1)^*\}$.

<u>PROOF</u>. For $n \equiv 3 \pmod 6$ a resolvable $B(\{3\},1;n)$ exists; completing such a design yields $n + (n-1)/2 \in \mathbb{B}\{4,((n-1)/2)^*\}$.
Writing $n = 6t + 3$ gives the lemma. □.

For $t = 2$ we get $22 \in \mathbb{B}\{4,7^*\}$.
For $t = 10$ we get $94 \in \mathbb{B}\{4,31^*\}$, and as soon as we know $31 \in \mathbb{B}\{4,7^*\}$ it follows that $94 \in \mathbb{B}\{4,7^*\}$.

(iii) Two orthogonal Latin Squares with Three Points Outside.

<u>LEMMA 3</u>. *If* $x \equiv 7$ or $43 \pmod{48}$ *then* $x \in \mathbb{B}\{4,7^*\}$.

<u>PROOF</u>. Let $x = 4t + 3$, then $t \equiv 1$ or $10 \pmod{12}$ and hence $t + 3 \in \mathbb{B}\{4\}$. Also $t \neq 2,6$ so $t \in T(4,1)$. Take a transversal design $T(4,1;t)$ on a set X and choose a fixed block $\{a_1,a_2,a_3,a_4\}$. Adjoin three new points x_0,x_1,x_2 to X and for each group G make a $B(\{4\},1;t+3)$ on each of the sets $G \cup \{x_0,x_1,x_2\}$, taking care that the design on the group containing a_i has $\{a_i,x_0,x_1,x_2\}$ as a block. Now remove the blocks $\{a_1,a_2,a_3,a_4\}$ and $\{a_i,x_0,x_1,x_2\}$ $(1\leq i\leq 4)$ and add the block $\{a_1,a_2,a_3,a_4,x_0,x_1,x_2\}$. This yields a $B(\{4,7^*\},1,x)$. □.

In particular we find 43, 55 and 91.

We still have to do 31,34,46,58,70,79,106 and 154.

(iv) The case x = 31 (found by A.E.B. and P.D.P. 11/45 in close cooperation). A Δ-factor of a graph is a 2-factor consisting of cycles of length 3. Or in design-theoretic terms: a Δ-factor is a parallel class of triples. Using this definition we clearly have:

<u>LEMMA 4</u>. $n \in \mathbb{B}\{4,7^*\}$ *iff there exists a design* B({3,4},1;n-7) *where the triples form* 7 *Δ-factors*. □.

In the current case we take for the set of vertices $X = Z_2 \times Z_2 \times Z_6$ (so that $|X|=24=31-7$), and the following blocks:

18 quadruples:

{(0,0,0),(0,1,0),(1,0,0),(1,1,0)}   mod (-,-,6)
{(0,0,0),(0,0,3),(1,1,1),(1,1,4)}   mod (2,2,-)
{(0,0,0),(0,0,4),(1,1,5),(0,1,2)}   mod (2,2,-)
{(0,0,1),(0,0,5),(1,1,2),(0,1,3)}   mod (2,2,-)

7 Δ-factors:

1. [{(0,0,0),(0,0,1),(0,0,2)}, {(0,0,3),(0,0,4),(0,0,5)}   mod (2,2,-)].

2,3. [{(0,0,0),(0,0,5),(0,1,1)}, {(0,0,2),(1,1,0),(0,1,3)},
{(1,1,1),(1,1,3),(1,0,4)}, {(0,0,4),(1,1,2),(1,0,5)}   mod (-,2,-)]
mod (2,-,-).

4,5. [{(0,0,2),(0,0,3),(1,0,4)}, {(1,1,2),(1,1,5),(0,1,1)},
{(0,0,0),(1,0,1),(0,1,4)}, {(1,1,0),(1,0,3),(0,1,5)}   mod (-,2,-)]
mod (2,-,-).

6,7. [{(0,0,0),(1,1,3),(0,1,5)}, {(0,0,2),(0,0,4),(1,0,0)},
{(0,0,1),(1,1,5),(1,0,4)}, {(0,0,3),(1,1,2),(1,0,1)}   mod (-,2,-)]
mod (2,-,-).

Clearly it is a finite task to check the correctness of this design.

(v) The Case x = 34.

Let $X = (Z_3 \times Z_9) \cup (I_2 \times Z_3) \cup \{x\}$, where the elements of $Z_3 \times Z_9$ are written (i,j) and those of $I_2 \times Z_3$ [i,j].

Take the following blocks:

$\{(i,j),(i+1,j+2),(i+2,j+2),(i+2,j+3)\}$: 27 blocks
$\{(i,j),(i+1,j+3),(i+1,j+5),[0,j-i]\}$: 27 blocks
$\{(i,j),(i+1,j+4),(i+1,j+8),[1,j]\}$: 27 blocks
$\{(i,j),(i,j+3),(i,j+6), \quad x\}$ $(j<3)$: 9 blocks.

(vi) The Case x = 46.

Let $X = (Z_3 \times Z_{13}) \cup (I_2 \times Z_3) \cup \{x\}$, and take the following blocks:

$\{(i,j+1),(i,j+3),(i,j+9),(i+1,j)\}$: 39 blocks
$\{(i,j+2),(i,j+6),(i,j+5),(i+1,j)\}$: 39 blocks
$\{(i,j),(i+1,j+1),(i+2,j+4),[0,i]\}$: 39 blocks
$\{(i,j),(i+1,j+2),(i+2,j+7),[1,i]\}$: 39 blocks
$\{(0,j),(1,j),(2,j), \quad x\}$: 13 blocks.

(vii) The Case x = 58.

Let $X = (Z_3 \times Z_{17}) \cup (I_2 \times Z_3) \cup \{x\}$, and take the following blocks:

$\{(i,j),(i,j+1),(i,j+4),(i+1,j+5)\}$
$\{(i,j),(i,j+2),(i,j+8),(i+1,j+11)\}$
$\{(i,j),(i,j+5),(i+1,j+2),(i+1,j+12)\}$
$\{(i,j),(i+1,j+8),(i+2,j+7),[0,i]\}$
$\{(i,j),(i+1,j+6),(i+2,j+4),[1,i]\}$
$\{(0,j),(1,j),(2,j), \quad x\}$.

(viii) The Cases 70 and 79.

In [7] Mills showed that $70 \in \mathbb{B}\{4,22^*\}$ and $79 \in \mathbb{B}\{4,13^*,22^*\}$. Since $13 \in \mathbb{B}\{4\}$ and $22 \in \mathbb{B}\{4,7^*\}$ it immediately follows that $\{70,79\} \subset \mathbb{B}\{4,7^*\}$.

(ix) The Cases 106 and 154.

LEMMA 5. *If* $t \in \mathbb{B}\{4,5,8,9,12,k^*\}$ *and* $3k+1 \in \mathbb{B}\{4,7^*\}$ *then* $3t+1 \in \mathbb{B}\{4,7^*\}$. *In particular this applies for* $k = 7$ or 11.

PROOF. Let $\mathcal{B}$ be a design on a set I_t with all block sizes congruent 0 or 1 (mod 4) but with one block of size 7 or 11. We can get a $B(\{4,7^*\},1;3t+1)$ on the set $I_t \times I_3 \cup \{x\}$ by taking for each block $B \in \mathcal{B}$ with $|B| \equiv 0$ or 1 (mod 4) a design $B(\{4\},1;\ 3|B|+1)$ on the set $B\times I_3 \cup \{x\}$, taking care that it contains the blocks $\{b\}\times I_3 \cup \{x\}$ for each $b \in B$; if B_0 is the block with $|B_0| \not\equiv 0$ or 1 (mod 4) then throw away all blocks $\{b\}\times I_3 \cup \{x\}$ for $b \in B_0$, and add the block $B_0\times I_3 \cup \{x\}$. We now have a $B(\{4,(3|B_0|+1)^*\},1;3t+1)$. Since $\{22,34\} \subset \mathbb{B}\{4,7^*\}$ this proves the lemma. □.

Now for x = 106 = 3.35 + 1 we take a resolvable $B(\{4\},1;28)$ and partially complete it with 7 points. (This is possible since it has (28-1)/3 = 9 parallel classes.) This yields a $B(\{4,5,7^*\},1;35)$ and we may apply the lemma. Likewise for X = 154 = 3.51 + 1 we take a resolvable $B(\{4\},1;40)$ and partially complete it with 11 points which yields a $B(\{4,5,11^*\},1;51)$ and we are through.

This completes the proof of our theorem.

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